OC540. Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where r is a rational number and n a positive integer. Find all triplets $(a, b, c) \in \mathbb{Q}_+ \times \mathbb{Q}_+ \times \mathbb{N}$ for which there exist infinitely many positive integers n satisfying $S_a(n) = (S_b(n))^c$

Originally from the 2012 Turkish IMO Team Selection Test, Day 3, Problem 7.

We received 5 correct and complete submissions. We present a solution submitted independently by two problem solving groups: UCLan Cyprus and Missouri State University.

We first observe that

$$\frac{S_r(n)}{n^{r+1}} = \frac{1}{n}\left(\left(\frac{1}{n}\right)^r + \left(\frac{2}{n}\right)^r + \cdots + \left(\frac{n}{n}\right)^r\right)$$

is a Riemann Sum of $f(x) = x^r$ over $[0, 1]$ with respect to the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n} \ldots, 1\}$. Since the mesh of the partition converges to 0 as n tends to infinity, we get that

$$\frac{S_r(n)}{n^{r+1}} \to \int_0^1 f(x)\,\mathrm{d}x = \frac{1}{r+1}.$$

Suppose now that $S_a(n) = S_b(n)^c$ for infinitely many values of n. Then

$$\frac{S_a(n)}{n^{a+1}} = \frac{S_b(n)^c}{n^{a+1}} = \frac{S_b(n)^c}{n^{c(b+1)}} n^{c(b+1)-(a+1)}$$

for infinitely many values of n. The left hand side converges to $1/(a+1)$. However, the right hand side converges to 0 if $a + 1 > c(b + 1)$, or $+\infty$ if $a + 1 < c(b + 1)$. So we must have $a + 1 = c(b + 1)$. In this case, the right hand side tends to $\frac{1}{(b+1)^c}$ so we must also have $a + 1 = (b + 1)^c$.

Now $c(b + 1) = (b + 1)^c$ gives $(b + 1)^{c-1} = c$. For $c > 1$, in order for $(b + 1)^{c-1}$ to be an integer, we must have that b is also an integer. But then $b \geqslant 1$ and so by Bernoulli's inequality $c = (b+1)^{c-1} \geqslant (1+1)^{c-1} \geqslant 1+(c-1) = c$. The inequality is strict if $c - 1 \geqslant 2$ so the only possibilities are $c = 1$ and $c = 2$.

For $c = 1$ we must have $a = b$ and for all such choices we have $S_a(n) = S_b(n)^c$ for all values of n. For $c = 2$ we must have $b + 1 = 2$ and therefore $b = 1$. Then we must also have $a + 1 = c(b + 1) = 4$ and therefore $a = 3$. In this case

$$S_a(n) = \left(\frac{n(n+1)}{2}\right)^2 = S_b(n)^2$$

for all values of n.

Therefore the only possible triples are $(a, a, 1)$ for $a \in \mathbb{Q}_+$ and $(3, 1, 2)$.